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AN ALTERNATIVE CONSTRUCTION OF THE $SU(2)$ CHERN-SIMONS PERTURBATION THEORY

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1. INTRODUCTION

In this note, we give an alternative construction of the 2-loop term of the Chern–Simons perturbation theory that is an invariant of a closed 3–manifold with a local system. This construction is deeply inspired by R. Bott and A. S. Cattaneo in [2], however our construction is more flexible than the original one. By using our construction, we gave a Morse theoretic description of the 2-loop term of the $SU(2)$ Chern–Simons perturbation theory in [5].

The Chern–Simons perturbation theory established by S. Axelrod and I. M. Singer in [1] and M. Kontsevich in [3] gives a topological invariant of a closed oriented 3-manifold with an acyclic local system. Let M be a closed oriented 3-manifold and let E be a local system on M . In the Chern–Simons perturbation theory, a propagator plays an important role. A propagator is a closed 2-form (with a twisted coefficient given by E) on the 2-point configuration space of M satisfying some conditions near the diagonal of $M \times M$. In this note, we mitigate these conditions.

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2. 2-LOOP TERM OF THE $SU(2)$ -CHERN–SIMONS PERTURBATION THEORY

In this section, we give a construction of 2-loop term of the Chern–Simons perturbation theory. This construction is deeply inspired by Bott and Cattaneo in [2].

Let M be a closed oriented 3-manifold, and let $\rho : \pi_1(M) \rightarrow SU(2)$ be an irreducible representation. The composition of ρ and the adjoint representation $SU(2) \rightarrow \text{Aut}(\mathfrak{su}(2))$ is an orthonormal representation of $\pi_1(M)$. We denote by $E = E_\rho$ the local system corresponding to this orthonormal representation. We assume that E is acyclic, namely

$$H^i(M; E) = 0,$$

for any i . For $x \in M$, we denote by E_x the object of E corresponding to $x \in M$.

For a submanifold B of a manifold A , we denote by $B\ell(A, B)$ the manifold obtained by a real blowing up of A along B , namely $B\ell(A, B) = (A \setminus B) \cup S\nu_B$. Here ν_B is the normal bundle of B and $S\nu_B$ is a unit sphere bundle of ν_B . Let $C_2(M) = B\ell(M^2, \Delta)$, where $\Delta = \{(x, x) \mid x \in M\} \subset M^2$ is the diagonal. $C_2(M)$ is a compactification of the configuration space $M^2 \setminus \Delta$. We denote by $q : C_2(M) \rightarrow M^2$ the blow down map. Since the normal bundle ν_Δ is isomorphic to the tangent bundle TM , we have $\partial C_2(M) = q^*(\Delta) \cong STM \cong M \times S^2$. Let $T : C_2(M) \rightarrow C_2(M)$ be the involution induced by the involution

$T_0 : M^2 \rightarrow M^2, (x, y) \mapsto (y, x)$. The involution T also acts on $\partial C_2(M) \cong STM$. We denote by $H_+^*(\partial C_2(M)), H_-^*(\partial C_2(M))$ the $+1, -1$ eigen space under the involution T_* , respectively.

Let $p_i : M^2 \rightarrow M$ be the projection, for $i = 1, 2$. Then $p_1^*E \otimes p_2^*E$ is a local system on M^2 and $F = q^*(p_1^*E \otimes p_2^*E)$ is a local system on $C_2(M)$.

The following lemma says how $SU(2)$ is special.

Lemma 2.1. $(E \otimes E)^- = 0$. In particular $H^2(\Delta; E \otimes E) = 0$.

Proof. The Lie bracket $[\cdot, \cdot]$ induces a local system morphism $b : E \otimes E \rightarrow E, f(x \otimes y) = [x, y]$. Since \mathfrak{su}_2 is semi-simple, $b : E_x \otimes E_x \rightarrow E_x$ is subjective for each $(x, x) \in \Delta$. Then we have $\dim(\ker b) = 6$. Since $b(T(x \otimes y)) = -b(x \otimes y)$, we have $(E_x \otimes E_x)^+ \subset \ker b$. On the other hand $\dim((E_x \otimes E_x)^+) = 6$. Then $(E \otimes E)^- \cong E$. Since $T_0|_\Delta = \text{id}$, $H_-^2(\Delta; E \otimes E) = H^2(\Delta; (E \otimes E)^-)$. Therefore $H^2(\Delta; (E \otimes E)^-) = H^2(\Delta; E) = 0$. \square

Let \mathbb{R} be the trivial local system. We define a local system morphism $c : \mathbb{R} \rightarrow E \otimes E$ as follows: For any $x \in M$, take an orthonormal basis $\{e_1^x, e_2^x, e_3^x\}$ of $E_x \cong \mathfrak{su}_2$. Then $\sum_{i=1,2,3} e_i^x \otimes e_i^x \in E_x \otimes E_x$ is independent of the choice of the basis. So we set $c(1) = \sum_{i=1,2,3} e_i^x \otimes e_i^x$.

Let $p : \partial C_2(M) \cong STM \cong M \times S^2 \rightarrow S^2$. Let $\omega_{S^2} \in \Omega^2(S^2; \mathbb{R})$ be a closed 2-form satisfying $\int_{S^2} \omega_{S^2} = 1$. Then $[p^*\omega_{S^2} - T^*p^*\omega_{S^2}]$ generates $^1 H_-^2(\partial C_2(M); \mathbb{R}) \cong H^2(S^2; \mathbb{R})$.

Definition 2.2. A closed 2-form $\omega \in \Omega^2(C_2(M); F)$ is said to be a *propagator* if there is a closed 2-form $\omega_{\mathbb{R}} \in \Omega^2(\partial C_2(M); \mathbb{R})$ and there is a closed 2-form $\eta \in \Omega^2(\Delta; E \otimes E)$ satisfy the following conditions:

- $[\omega_{\mathbb{R}}] = \frac{1}{2}[p^*\omega_{S^2} - T^*p^*\omega_{S^2}]$,
- $T^*\omega_{\mathbb{R}} = -\omega_{\mathbb{R}}, T^*\eta = -\eta$, (of course, $T^*\omega = -\omega$)
- $\omega|_{\partial C_2(M)} = c_*\omega_{\mathbb{R}} + q^*\eta$.

Lemma 2.3. For any M and E , there is a propagator.

Proof. Thanks to Lemma 2.1, in the following cohomology long exact sequence of the pair $(C_2(M), \partial C_2(M))$, $H_-^3(C_2(M), \partial C_2(M); F) \cong H_-^3(M^2, \Delta; p_1^*E \otimes p_2^*E) \cong H_-^2(\Delta; E \otimes E) = 0$.

$$\dots \rightarrow H_-^2(C_2(M); F) \rightarrow H_-^2(\partial C_2(M); F) \rightarrow H_-^3(C_2(M), \partial C_2(M); F) \rightarrow \dots$$

Then for any $\omega_{\mathbb{R}}$ and η , there is a propagator. \square

We define a local system morphism $\text{Tr} : E \otimes E \otimes E \rightarrow \mathbb{R}$ by $\text{Tr}(x \otimes y \otimes z) = \langle [x, y], z \rangle$. Here $[\cdot, \cdot]$ is the Lie bracket and $\langle \cdot, \cdot \rangle$ is the Killing form.

Let ω be a propagator that $\omega|_{\partial C_2(M)} = c_*\omega_{\mathbb{R}} + q^*\eta$.

Definition 2.4.

$$\begin{aligned} Z_{\Theta}(\omega) &= \int_{C_2(M)} \text{Tr} \omega^3 \in \mathbb{R}, \\ Z_{O-O}(\omega) &= \int_{C_2(M)} \text{Tr}(\omega \wedge \eta_1 \wedge \eta_2) \in \mathbb{R}, \\ Z(\omega) &= Z_{\Theta}(\omega) - 3Z_{O-O}(\omega). \end{aligned}$$

Here $\eta_1 = q^*p_1^*\eta$ and $\eta_2 = q^*p_2^*\eta$.

¹It is easy to check that the cohomology class $[p^*\omega_{S^2} - T^*p^*\omega_{S^2}]$ is independent from the choice of isomorphism $STM \cong M \times S^2$.

Let X be a closed oriented 4-manifold such that the Euler characteristic of X is zero and $\partial X = M$. Take a sub \mathbb{R}^3 bundle $T^v X$ of TX satisfying $T^v X|_M = TM$. Let $ST^v X$ be the unit sphere bundle of $T^v X$. Let F_X be the tangent bundle along the fiber of the sphere bundle $ST^v X \rightarrow X$. Take a closed 2-form $\omega_X \in \Omega^2(ST^v X; \mathbb{R})$ such that:

- $[\omega_X] = \frac{1}{2}e(F_X)$ where $e(F_X)$ is the Euler class of F_X ,
- $\omega_X|_{STM} = \omega_{\mathbb{R}}$ under the identification $STM \cong \partial C_2(M)$.

Lemma 2.5 (Proposition 5.3 in [4]). $\int_{ST^v X} \omega_X^3 - \frac{3}{4}\text{Sing}X$ is independent of the choice of X and ω_X .

Definition 2.6. $I(\omega) = \int_{ST^v X} \omega_X^3 - \frac{3}{4}\text{Sing}X$.

Theorem 2.7. $Z(M; E) = Z(\omega) - 6I(\omega) \in \mathbb{R}$ is an invariant of (M, E) .

Remark 2.8. This invariant or similar invariants were given by Kontsevich in [3], Axelrod, Singer in [1], Bott and Cattaneo in [2]. They used more limited propagators, in particular the restriction of propagators to $\partial C_2(M)$ were written by using a framing of M . They used a signature defect of the framing instead of our $I(\omega)$.

3. PROOFS

Let ω, ω' be propagators such that $\omega|_{\partial C_2(M)} = c_*\omega_{\mathbb{R}} + q^*\eta$ and $\omega'|_{\partial C_2(M)} = c_*\omega'_{\mathbb{R}} + q^*\eta'$.

Lemma 3.1. $[\eta] = [\eta'] = 0, [\omega_{\mathbb{R}}] = [\omega'_{\mathbb{R}}]$.

Proof. This is a direct consequence of the condition about propagators and Lemma 2.1. \square

Lemma 3.2. If $\omega|_{\partial C_2(M)} = \omega'|_{\partial C_2(M)}$, then $Z(\omega) = Z(\omega')$.

Proof. $Z_{\Theta}(\omega) - Z_{\Theta}(\omega') = \int_{C_2(M)} \text{Tr}((\omega - \omega')(\omega^2 + \omega\omega' + (\omega')^2)) = (*)$. Since $\omega|_{\partial C_2(M)} = \omega'|_{\partial C_2(M)}$, $[\omega - \omega'] \in H^2(C_2(M); F)$. Thanks to Lemma 2.1, in the following cohomology long exact sequence of the pair $(C_2(M), \partial C_2(M))$, we have $H_-^2(C_2(M), \partial C_2(M); F) \cong H_-^2(M^2, \Delta; p_1^*E \otimes p_2^*E) = H_-^1(\Delta; E \otimes E) = 0$.

$$\dots \rightarrow H_-^2(C_2(M), \partial C_2(M); F) \rightarrow H_-^2(C_2(M); F) \rightarrow H_-^2(\partial C_2(M); F) \rightarrow \dots$$

This implies that $[\omega - \omega'] = 0 \in H^2(C_2(M); F)$. Therefore $(*) = 0$.

It is easy to see that $Z_{O-O}(\omega) = \int_{C_2(M)} \text{Tr}(\omega\eta_1\eta_2) = \int_{C_2(M)} \text{Tr}(\omega'\eta_1\eta_2)$. \square

Proposition 3.3. If $\omega_{\mathbb{R}} = \omega'_{\mathbb{R}}$, then $Z(\omega) = Z(\omega')$.

Proof. There is a cochain $\xi \in \Omega_-^1(\Delta; E \otimes E)$ such that $d\xi = \eta'$. Thanks to Lemma 3.2, it is enough to show that the case $\omega' = \omega + d\xi$ and $\eta = 0$.

$$\begin{aligned}
Z_\Theta(\omega') - Z_\Theta(\omega) &= \int_{C_2(M)} \text{Tr}((\omega + d\xi)^3 - \omega^3) \\
&= \int_{C_2(M)} \text{Tr}(3\omega^2 d\xi + 3\omega(d\xi)^2 + (d\xi)^3) \\
&= \int_{\partial C_2(M)} \text{Tr}(3(c_*\omega_{\mathbb{R}})^2 \xi + 3\xi c_*\omega_{\mathbb{R}} d\xi + \xi(d\xi)^2) \\
&= \int_{\partial C_2(M)} \text{Tr}(3\xi c_*\omega_{\mathbb{R}} d\xi) \\
&= 3 \int_{\Delta} \text{Tr}(\xi d\xi).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
Z_{O-O}(\omega') - Z_{O-O}(\omega) &= \int_{C_2(M)} \text{Tr}((\omega + d\xi)\eta_1\eta_2) \\
&= \int_{C_2(M)} \text{Tr}(\omega q^* p_1^* d\xi \eta_2 + d\xi \eta_1 \eta_2) \\
&= \int_{\partial C_2(M)} \text{Tr}(c_*\omega_{\mathbb{R}} \xi \eta + \xi \eta^2) \\
&= \int_{\Delta} \text{Tr}(\xi d\xi).
\end{aligned}$$

Therefore $Z_\Theta(\omega) - 3Z_{O-O}(\omega) = Z_\Theta(\omega') - 3Z_{O-O}(\omega)$. \square

Proof of Theorem 2.7. It is enough to show that the case of $\eta = \eta' = 0$. Thanks to Lemma 3.1, there is a 1-form $\nu \in \Omega^1(\partial C_2(M); \mathbb{R})$ such that $\omega'_{\mathbb{R}} - \omega_{\mathbb{R}} = d\nu$. Thanks to Lemma 3.2, we can assume that $\omega' = \omega + c_* d\nu$.

$$\begin{aligned}
Z_\Theta(\omega') - Z_\Theta(\omega) &= \int_{C_2(M)} \text{Tr}(c_* d\nu (\omega^2 + \omega\omega' + (\omega')^2)) \\
&= \int_{\partial C_2(M)} \text{Tr}(c_*(1) \otimes c_*(1) \otimes c_*(1)) \nu (\omega_{\mathbb{R}}^2 + \omega_{\mathbb{R}} \omega'_{\mathbb{R}} + (\omega'_{\mathbb{R}})^2) \\
&= 6 \int_{\partial C_2(M)} \nu (\omega_{\mathbb{R}}^2 + \omega_{\mathbb{R}} \omega'_{\mathbb{R}} + (\omega'_{\mathbb{R}})^2).
\end{aligned}$$

$$Z_{O-O}(\omega') - Z_{O-O}(\omega) = 0 - 0 = 0.$$

We can take $\omega'_X = \omega_X + d\nu$. Then we have

$$\begin{aligned}
I(\omega') - I(\omega) &= \int_{ST^v X} d\nu (\omega_X^2 + \omega_X \omega'_X + (\omega'_X)^2) \\
&= \int_{STM} \nu (\omega_{\mathbb{R}}^2 + \omega_{\mathbb{R}} \omega'_{\mathbb{R}} + (\omega'_{\mathbb{R}})^2).
\end{aligned}$$

Therefore $Z(\omega) - 6I(\omega) = Z(\omega') - 6I(\omega')$. \square

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